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ON AN EXACT SOLUTION OF THE PROBLEM OF ELASTIC WAVE DIFFRACTION BY A WEDGE

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The exact solution is obtained for the nonstationary problem of plane longitudinal and transverse elastic wave diffraction by a wedge with the following boundary conditions: normal stresses and tangential displacements are zero.

The problem of elastic wave diffraction by a rigid wedge imbedded in an infinite elastic medium without friction has been considered in [1]. Obtaining a closed solution in this case is possible because the boundary conditions for the longitudinal and transverse potentials are separated (until conditions on the edge are taken into account). It has been clarified [2] that in investigating the interaction between elastic waves and plane boundaries the boundary conditions for the potentials are still separated even when the normal stresses and tangential displacements are given on the boundary.

1. Formulation of the problem. An elastic medium with propagation velocities a and b for the longitudinal and transverse waves fills the exterior of a wedge, on whose boundaries the conditions that the radial displacement and shear stress vanish, are given.

The connection between the radial u_r and tangential u_θ displacements and the longitudinal φ and transverse ψ potentials is given by the relationships

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi}{\partial r}$$

The boundary conditions for the potentials on the wedge faces are the following:

$$\varphi = 0, \quad \partial\psi / \partial\theta = 0 \quad \text{for} \quad \theta = 0, \gamma \quad (\gamma = k^{-1}\pi, \ k < 1)$$

where γ is the external angle of the wedge.

The conditions on the edge are taken in the same form as in [1], namely, it is required that the displacements be bounded and that the stresses and strains grow more slowly than r^{-1}. It is assumed that the incident wave potential is described by the Heaviside step function.

The wave fronts being formed upon incidence of a longitudinal wave on the wedge are shown in Fig. 1 for the cases (a) when no shadow domain is formed (the angle of incidence is $\beta > \gamma - 3\pi / 2$) and (b) when a shadow region is formed ($\beta < \gamma - 3\pi / 2$). The lines abc and egh are the fronts of the diffracted longitudinal and transverse waves. The coefficient of incident longitudinal wave reflection is -1 and of the transverse wave is 1.

2. Longitudinal wave incidence. In the interior part of the region bounded by the wedge faces oa and oc as well as the diffracted longitudinal wave front abc, we introduce the independent variables $\varepsilon_1 = r_1^{-1} - \sqrt{r_1^{-2} - 1}$ and θ, where $r_1 = r\,(at)^{-1}$. Then the wave equation for the longitudinal potential goes over into the Laplace equation.

The domain under consideration is mapped conformally into an upper semicircle of the y_1 plane by means of the following transformation:

$$y_1 = R_1 e^{i\nu} \qquad (R_1 = \varepsilon_1{}^k, \ \nu = k\theta)$$

The analytic function $W_a(y_1) = \varphi_a(y_1) + if_1(y_1)$ of the complex variable y_1 is introduced in this region ($\varphi_a(y_1)$ is the longitudinal potential corresponding to the acoustic solution). By using the symmetry principle, the function $W_a(y_1)$ can be continued analytically into the lower part of the unit circle.

Therefore, the problem is formulated as follows: find the real part of the function $W_a(y_1)$ under the following conditions on the surface of the unit circle ($R_1 = 1$):

$$\operatorname{Re} W_a = \begin{cases} 0, & -\nu_1 < \nu < \nu_1, \quad \nu_2 < \nu < 2\pi - \nu_2 \\ 1, & \nu_1 < \nu < \nu_2 \\ -1, & 2\pi - \nu_2 < \nu < 2\pi - \nu_1 \end{cases}$$

$$(\nu_1 = k\theta_1, \ \nu_2 = k\theta_2, \ \theta_1 = \pi / 2 - \beta)$$

$$\theta_2 = \begin{cases} 2\gamma - 3\pi / 2 - \beta, & \beta > \gamma - 3\pi / 2 \\ 3\pi / 2 + \beta, & \beta < \gamma - 3\pi / 2 \end{cases}$$

The solution of the Dirichlet problem for a circle is known [3].

However, the acoustic solution obtained will not generally satisfy the condition on the edge. In order to satisfy the condition on the edge, let us represent the longitudinal potential as the sum of two functions. The first function describes the acoustic solution. The second is selected so that it would satisfy the Laplace equation, the zero boundary conditions on the real axis of the y_1 plane, and would permit compliance with the condition on the edge.

Therefore, the expression for the longitudinal potential can be written as

$$\varphi(r_1, \theta) = \varphi_a(r_1, \theta) + \operatorname{Re}[ia_1(y_1 + y_1^{-1})] \tag{2.1}$$

where a_1 is a still unknown coefficient, which should be determined from the condition

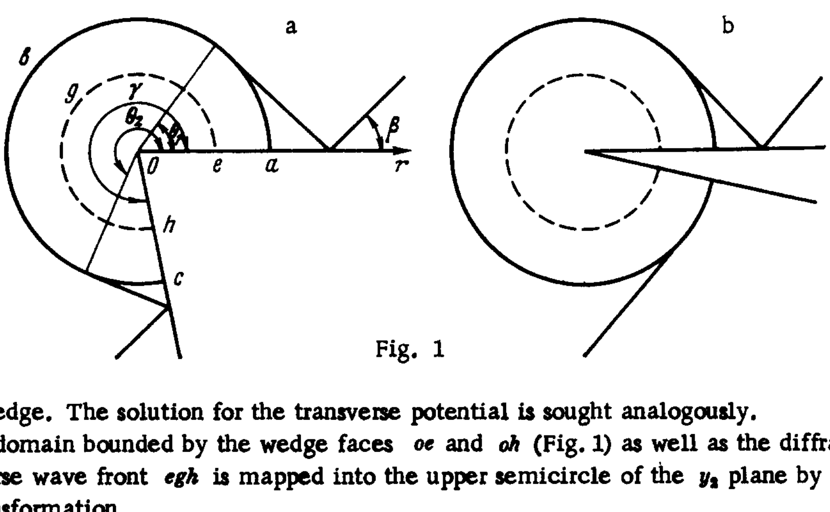

Fig. 1

on the edge. The solution for the transverse potential is sought analogously.

The domain bounded by the wedge faces oe and oh (Fig. 1) as well as the diffracted transverse wave front egh is mapped into the upper semicircle of the y_2 plane by using the transformation

$$y_2 = R_2 e^{i\nu} \qquad R_2 = \left(r_2^{-1} - \sqrt{r_2^{-2} - 1}\right)^k, \quad r_2 = r\,(bt)^{-1}$$

The function $W_2(y_2) = \psi(y_2) + if_2(y_2)$ is introduced. This function is found so that it would satisfy the Laplace equation, the conditions

$$\operatorname{Re} W_2 = 0 \quad \text{for} \quad R_2 = 1,\ 0 < \nu < \pi$$
$$\operatorname{Re} \partial W_2/\partial\theta = 0 \quad \text{for} \quad \nu = 0,\ \nu = \pi$$

and would permit compliance with the boundary conditions on the edge. Then it is possible to write

$$W_2(y_2) = a_2 (y_2 - y_2^{-1}) \tag{2.2}$$

where a_2 is a still unknown coefficient.

Using (2.1) and (2.2), we write the expressions for the displacements in the domain bounded by the wedge faces and the diffracted transverse wave front as

$$u_r = 2k\left(\pi r\sqrt{1-r_1^2}\right)^{-1}\left[\operatorname{Re}(iQ) - {}^1/_2 a_1\pi (R_1 + R_1^{-1})\sin\nu\right] - a_2 k r^{-1}(R_2 - R_2^{-1})\sin\nu \tag{2.3}$$
$$u_\theta = 2k/\pi r\left[\operatorname{Re}(-Q) - {}^1/_2 a_1\pi (R_1 - R_1^{-1})\cos\nu\right] - a_2 k r^{-1}(1 - r_2^2)^{-1/2}(R_2 + R_2^{-1})\cos\nu$$
$$Q = y_1\left[(\cos\nu_2 - y_1)(1 + y_1^2 - 2y_1\cos\nu_2)^{-1} - (\cos\nu_1 - y_1)(1 + y_1^2 - 2y_1\cos\nu_1)^{-1}\right]$$

Performing the asymptotic expansions for (2.3) as $r \to 0$ and using the conditions of boundedness of the displacements on the wedge edge, we obtain the following dependences for the coefficients a_1 and a_2:

$$a_1 = 2\pi^{-1}\left[1 + (ab^{-1})^{2k}\right]^{-1}(\cos\nu_1 - \cos\nu_2), \quad a_2 = a_1 (ab^{-1})^k \tag{2.4}$$

Taking account of (2.4), the expressions for the potentials are written as

$$\varphi = \varphi_a(r, \theta, t) - 4\pi^{-1}\left[1 + (ab^{-1})^{2k}\right]^{-1}(R_1 - R_1^{-1})\sin\pi k \sin k(\pi/2 + \beta)\sin k\theta$$
$$\psi = 4\pi^{-1}\left[1 + (ab^{-1})^{2k}\right]^{-1}(ab^{-1})^k (R_2 - R_2^{-1})\sin\pi k \sin k(\pi/2 + \beta)\cos k\theta$$

In contrast to the case of longitudinal wave incidence on a rigid wedge imbedded without friction in an elastic medium, no such value of the angle of incidence exists for which the solution would agree with the acoustic solution in the case under consideration. Exactly as in [1], it can be seen that the elastic terms and the acoustic terms have the identical intensity both near the wedge edge and near the diffracted wave fronts. The acoustic and longitudinal components of the radial displacement are of the order of $(r_1^{-1} - 1)^{-1/2}$ in the neighborhood of the diffracted longitudinal wave, and the tangential displacement components are of the order of unity.

Near the transverse wave front both the radial displacement components are of the order of one, and the tangential displacments are of the order of $(r_1^{-1} - 1)^{-1/2}$.

3. Transverse wave incidence. In this case the solution can be obtained by the same method as in Sect. 2. For example, let us present the expressions for the potentials (no shadow region)

$$\varphi = 4\pi^{-1}\left[1 + (ab^{-1})^{2k}\right]^{-1}(ab^{-1})^k (R_1 - R_1^{-1})\sin\pi k\cos k(\pi/2 + \beta)\sin k\theta$$
$$\psi = \psi_a(r, \theta, t) - 4\pi^{-1}\left[1 + (ba^{-1})^{2k}\right]^{-1}(R_2 - R_2^{-1})\sin\pi k\cos k(\pi/2 + \beta)\cos k\theta$$

Here $\psi_a(r, \theta, t)$ is the solution of the acoustic problem in the domain bounded by the wedge faces (on which $\partial\psi_a/\partial\theta = 0$) and the diffracted transverse wave front. The solution agrees with the acoustic solution when the incident ray is directed along the wedge bisectrix. As in Sect. 2, the additional elastic terms are of the same order as the acoustic terms.

Let us note that as the shear modulus of the medium surrounding the wedge tends to zero, the solution of the problem under consideration goes over into the solution of the problem of acoustic wave diffraction by a hollow wedge.

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